Generalization of Path-Integration to a Complex Time-Discretization Index

Abstract:

We generalize the path-integration process in quantum mechanics to the case of a complex-valued time-discretization index. As is well known, the propagator for particle movement in the path integral formulation is given as a limit of a multiple integral expression, with a time-slicing index approaching infinity. By allowing for a complex-valued index (a complex *number* of time slices), and augmenting the formula for the action—so as to maintain the status of the path contribution formula as a phase factor—the limit is indeterminate and yields a variety of different results for the propagator. These different propagators are postulated to apply to different universes in an expansive multiverse. This process is completely different from the standard practice of calculating path integrals in imaginary time, as it is the time-slicing index, as opposed to the time interval, that is generalized to the complex numbers; the results (summarized in Fig. 1 and equations (7)-(12)) are novel as well.

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I. Introduction and Central Idea

A. Overview

In the path integral formulation of quantum theory, the probability amplitude that a particle will travel from point A to point B over a time interval T is proportional to a functional integral of a contribution formula over all possible paths connecting A and B. Such a functional integral is frequently defined via a time-discretization procedure, whereby the time interval T is divided into an index N number of "slices," and the contribution formula is integrated over all "discrete paths" formed by the slicing process. Such a discrete path is given by straight lines within each slice, connecting the spatial coordinates x_i located at each slice point. Thus, an integral over these discrete paths is given as an integral over the x_i . Allowing the index N to approach infinity effects the functional integral over all possible paths. Therefore, the propagator is proportional to [2]

$$P(a,b) \cong \lim_{n\to\infty} \iiint \dots \int \quad exp\left(\frac{i}{\hbar}S\right) dx_1 \dots dx_{n-1} \quad (1)$$

In this case, the contribution formula is given as a phase factor dependent on the action. As Feynman has noted [2], contributions from different paths differ only in phase, not magnitude. The action itself is given as the integral of the Lagrangian over the time-interval T. Indeed,

$$S = \int_0^T L\,dt \quad (2)$$

Here, L is the Lagrangian over time. Often, in physical problems, this integral over time is divided into N different integrals corresponding to each time slice. In general, the Lagrangian L is given as

$$L = \frac{m}{2}\mathrm{v}^{\mathbf{2}} - \mathrm{V} \quad (3),$$

where V is the potential, and the velocity v is, between t_{i-1} and t_i,

$$\mathrm{v} = \frac{(x_i - x_{i-1})}{T}N \quad (4)$$

The index N is typically taken to be a positive integer. While the study of path integrals in imaginary time (that is, for imaginary time interval T) is well known, the index N describing the time-discretization structure has always been taken to be a natural number. After all, it is difficult to imagine a non-integer "number" of time-slices. The central idea behind this paper is to generalize the index N to the complex numbers. We utilize an analytic continuation to determine the result. In the process, a multiple integral over a complex "number" N of variables is taken, and subsequently a limit as N approaches infinity is introduced. As we shall demonstrate, this limit is indefinite, and has a spectrum of possible results depending upon the way N goes to infinity in the complex plane. These different results are postulated to apply to different universes.

In the case of complex N, the quantity (4), specifying the particle velocity, is complex valued. Therefore, the Lagrangian, as in (3), is also complex, and apparently so is the action (2). Therefore, the contribution formula as it appears in (1) (the integrand), is not only a phase factor, but possesses a real exponential part. This contingency is in violation of Feynman's postulate [2], that the contribution of different paths differ only in phase and not in magnitude. Thus, if we are to generalize (1) to complex-valued N, we must augment the formulas for path contribution (integrand of (1), delineated in (2)-(4)) to maintain the status of the path integrand as a phase factor.

Thus, to keep the action real-valued, we re-define (2) as a magnitude,

$$S = \left|\int_0^T L\,dt\right| \quad (5)$$

The lagrangian L is still given as in (3); however, its complex value (dependent on complex N) no longer disrupts the real nature of the action—thus, under suitable interpretation, the contribution formula remains a phase factor (although there are still a complex number (N) *of variables* in (1), that appear in (5)).

Thus, the contribution formula is given as

$$exp\left(\frac{i}{\hbar}\left|\int_0^T L\,dt\right|\right) \quad (6)$$

thus yielding the standard action in the case of real N[1].

The presence of the magnitude in (6) makes the limit (1), as applied to complex-valued N, indeterminate. Indeed, many complex-valued functions involving a magnitude, for example $^{z}/_{|z|}$, yield indeterminate results as the complex variable approaches infinity. In the case of (1), as we shall show, a spectrum of different results are obtained depending upon how the complex-valued index N approaches infinity in the plane. These different propagators are postulated to apply to different universes.

This generalization may seem arbitrary. However, similar such generalizations have found utility in fundamental physics. For example, an extension of the Einstein field equations to the case of a nonzero cosmological constant led to an anthropic multiverse argument, with different universes corresponding to different values of the constant [7]. Additionally, the well-known landscape concept in superstring theory was a result of postulating the existence of different universes corresponding to different Calabi-Yau manifolds [4], and similar generalizations of physical law have also found support as potentially implying the existence of a multiverse [11]. Mathematically, the use of fractional and complex-order differentiation, roughly analogous to the notion of functional integration with a complex discretization index, has found great application in physics [12]. Similarly, this paper sees a generalization of the path-integral time-discretization process to the case of complex valued N. To keep the generalization as natural as possible, we maintain the status of the contribution formula as a pure phase factor. By taking the magnitude of the action, the path integrand remains a phase factor (*under suitable interpretation*—it is not quite clear how to directly quantify the value of the integrand of (1) that may contain a complex number N of variables). The multitude of results depending upon the way by which the index N approaches infinity in the complex plane are interpreted to apply to different universes. This generalization may lead to some advances in the field.

B. Further Generalization of the Contribution Factor

As currently construed, the present construction involves two essential generalizations of the Feynman path integral: firstly, an extension of the time-discretization index N to complex values, and secondly, a corresponding generalization of the path contribution formula, preserving its status as a phase

[1] This formula reduces to the standard contribution formula for N real. As delineated in footnote (3), in the case of potential V, the contribution formula is best defined as a continuation of a power series.

factor. However, the continuation (6) to complex N is only one such possible generalization. Applying the magnitude to the action is not the only process yielding an analytic continuation of the contribution factor. In the complex plane, a variety of magnitudes exist; the Lebesgue p-norm summarizes most useful magnitudes, including the well-known "taxicab" and "maximum" norms. In order to use the most complete generalization, we consider different continuations of the contribution formula, and replace the magnitude in (5) with a general p-norm.

On a complex number $x + iy$, the p-norm is defined as [5]

$$|x + iy|_p = (|x|^p + |y|^p)^{1/p}$$

As such, we adjust (6) to become

$$exp\left(\frac{i}{\hbar}\left|\int_0^T L\,dt\right|_p\right) \quad (7)$$

thus capturing the different analytic continuations that may be applied to the contribution formula, subject to the phase-factor requirement.

II. Free Particle Calculation

We shall utilize this generalization to compute the probability amplitude corresponding to a free particle. Using (1), (7), we obtain

$$K_0 \cong \lim_{n\to\infty} \iiint \dots \int \quad exp\left(\frac{i}{\hbar}\left|\int_0^T L\,dt\right|_p\right) dx_1 \dots dx_{n-1}$$

We may divide the integral from 0 to T, that defines the action, into n smaller integrals corresponding to each "time slice." Following through with this procedure, we deduce

$$\lim_{n\to\infty} \iiint \dots \int \quad exp\left(\frac{i}{\hbar}\frac{m}{2T^2}\left|\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} \left\{x_{j+1} - x_j\right\}^2 n^2\,dt\right|_p\right) dx_1 \dots dx_{n-1}$$

Clearly, assuming that the real or imaginary part of a sum is the sum of the real or imaginary parts of the terms,

$$Re\left(\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} \left\{x_{j+1} - x_j\right\}^2 n^2\,dt\right) = Re(n)\left[\sum_{j=0}^{n-1} T\left\{x_{j+1} - x_j\right\}^2\right]$$

Likewise,

$$Im\left(\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} \left\{x_{j+1} - x_j\right\}^2 n^2\,dt\right) = Im(n)\left[\sum_{j=0}^{n-1} T\left\{x_{j+1} - x_j\right\}^2\right]$$

Implying that

$$\left|\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}}\{x_{j+1}-x_j\}^2\, n^2\, dt\right|_p = |n|_p\left[\sum_{j=0}^{n-1}T\{x_{j+1}-x_j\}^2\right]$$

Thus, this integral simplifies to[2]

$$\lim_{n\to\infty}\iiint\dots\int\ \ exp\left(A\sum_{j=0}^{n-1}\{x_{j+1}-x_j\}^2\right)dx_1\dots dx_{n-1}$$

Here, A is given by

$$A=\frac{i}{\hbar}\frac{m}{2}\frac{|n|_p}{T}$$

We now introduce a change of variables, substituting z_j for $x_j - x_{j-1}$. Thus, we have

$$\lim_{n\to\infty}\iiint\dots\int\ \ exp\left(A\left\{\sum_{j=1}^{n-1}\{z_j\}^2+\{x_n-x_{n-1}\}^2\right\}\right)dz_1\dots dz_{n-1}$$

Simplifying,

$$\lim_{n\to\infty}\iiint\dots\int\ \ exp\left(A\left\{\sum_{j=1}^{n-1}\{z_j\}^2+\left(x_n-x_0-\sum_{i=1}^{n-1}z_i\right)^2\right\}\right)dz_1\dots dz_{n-1}$$

Use of Fourier integration to decompose the squared exponential produces

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}exp\,\left(\frac{k^2}{4A}\right)\iiint\dots\int\ \ exp\left(A\sum_{j=1}^{n-1}\{z_j\}^2\right)exp\left(ik\left(x_n-x_0-\sum_{i=1}^{n-1}z_i\right)\right)dz_1\dots dz_{n-1}\,dk$$

Separation of variables may now be utilized to compute this multiple integral. Specifically, we calculate

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}exp\,\left(\frac{k^2}{4A}\right)exp\big(ik(x_n-x_0)\big)\left\{\int\ \ exp(Az^2-ikz)dz\right\}^{n-1}\ dk$$

This yields

$$\lim_{n\to\infty}exp\left(\frac{mX^2i\ \ |n|_p}{2\hbar Tn}\right)$$

In which X is the total displacement traversed by the particle $(x_n - x_0)$. It is at this point that we ascertain the limit of our complex-valued index n. Specifically, we introduce a substitution of variables, by writing the index in polar form,

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[2] Here, again, we assume that the real or imaginary part of a sum is equal to the sum of the corresponding real or imaginary parts, even in the case of complex n. We thus generalize the known properties of the real and imaginary parts to the case of a sum with a complex "number" of terms. The resulting sum, although real-valued, depends on a complex number of distinct variables, and thus its value is not well-defined. Similarly, although the Dirac delta "function" is not well-defined over its domain, its integral is; even though it is impossible to quantify the value of a function of a complex number of variables, we may use a separation of variables to assign a value to its integral.

$$n = r\exp(it)$$

Here, r is a real number which approaches infinity as n approaches infinity, while t is a separate real parameter. Thus, we have

$$\lim_{r\to\infty} exp\left(\frac{mX^2 i\, r^2\left(|\exp(it)|_p\right)}{2\hbar T r^2 exp(it)}\right) = K_{0\,std}^{C} \quad (8)$$

In which $K_{0\,std}$ is the standard probability amplitude for a free particle, while C is given by

$$\frac{|\exp(it)|_p}{exp(it)} \ (9)$$

C parameterizes a region in the complex plane, located between the parametric curves $\big((|\cos(t)| + |\sin(t)|)\cos(t), -(|\cos(t)| + |\sin(t)|)\sin(t)\big)$ and $(max(|\cos(t)|, |\sin(t)|)\cos(t), -max(|\cos(t)|, |\sin(t)|)\sin(t))$, $\pi \le t \le 2\pi$, for $p \ge 1$. An image of this figure is provided in **Fig.1**.

Thus, this generalization of the time-discretization index, subject to appropriate definitions of the action in order to yield a phase factor contribution formula, yields a variety of different probability amplitudes, due to the non-uniqueness of the limit. These probability amplitudes are complex powers of the standard kernel, with the exponent defined within a specific region of the complex plane. Alternatively, these particles could be viewed as behaving with a complex mass, with this complex mass being a multiplier of the normal mass, the multiplier appearing within the specific area illustrated in **Fig. 1**.

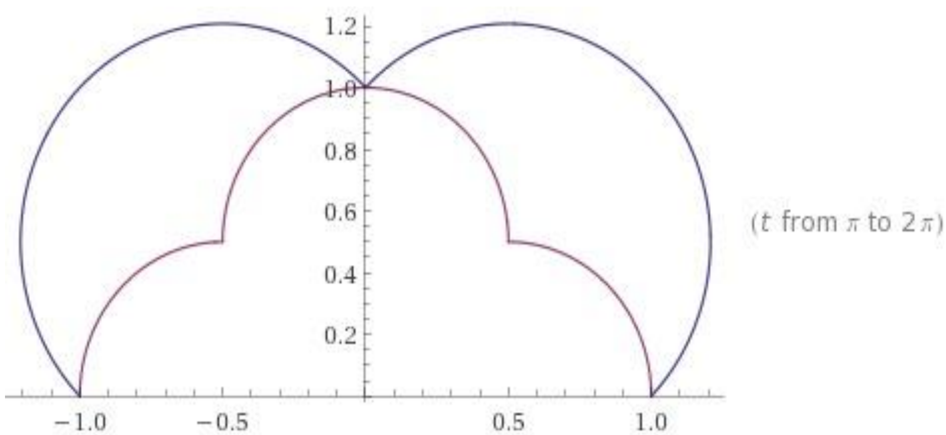

Fig. 1- Contour in the Complex Plane which Delineates Different Probability Amplitudes in Generalization

III. The Particle in a Potential

A. Perturbation Expansion

We shall apply perturbation methods to calculate the path integrals associated with a potential, and analytically extend the results to the case of a complex number of time slices. We begin by noting that the integral of the potential may be expanded as

$$\int V\,dt = \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} V(x)dt = \sum_{i=0}^{n-1} \frac{T}{n} \int_0^1 V(x_j + (x_{j+1} - x_j)s)ds$$

As n approaches infinity (we will assume a potential only explicitly dependent on position, not time), this expression provides the integral of the potential. Given this expression, we consider the contribution formula (7), and simplify[3]:

$$exp\left(\frac{i}{\hbar}\left|\int_0^T L\,dt\right|_p\right) =$$

$$exp\left(\frac{i}{\hbar}\left(\left|Re(n)\left[\sum_{j=0}^{n-1}\frac{m}{2}\frac{\{x_{j+1}-x_j\}^2}{T}\right] - TRe\left[\frac{1}{n}\right]\sum_{j=0}^{n-1}\int_0^1 V(x_j + (x_{j+1}-x_j)s)ds\right|^p\right.\right.$$

$$\left.\left. + \left|Im(n)\left[\sum_{j=0}^{n-1}\frac{m}{2}\frac{\{x_{j+1}-x_j\}^2}{T}\right] - TIm\left[\frac{1}{n}\right]\sum_{j=0}^{n-1}\int_0^1 V(x_j + (x_{j+1}-x_j)s)ds\right|^p\right)^{1/p}\right)$$

Inserting this formula into (1) in the case of non-zero potential V, expanding the contribution formula in a Taylor series in $-\sum_{j=0}^{n-1}\int_0^1 V(x_j + (x_{j+1}-x_j)s)ds$, and simplifying, we obtain

$$\lim_{n\to\infty}\left(K - \frac{i}{\hbar}B\int\int\ ...\int\ exp\left(\frac{i}{\hbar}S[0]\right)\left(T\sum_{j=0}^{n-1}\int_0^1 V(x_j + (x_{j+1}-x_j)s)ds\right)dx_1 \ldots dx_{n-1} + \cdots\right)$$

In which the ellipsis denotes the sum of the other terms, S[0] is the free particle action, and B is

$$B = \left(Re(n)Re\left(\frac{1}{n}\right)|Re(n)|^{p-2} + Im(n)Im\left(\frac{1}{n}\right)|Im(n)|^{p-2}\right)(|Re(n)|^p + |Im(n)|^p)^{\frac{1}{p}-1}$$

To derive an analogue of the Schrodinger equation, as in 3.B, we need only consider infinitesimal time-translations T. For the limit of small T, as in 3.B, we have

[3] Technically, as the derivative of the exponential is discontinuous in V, we should define the path contribution formula as an analytic continuation to the case that $V \neq 0$. This analytic continuation still yields the standard contribution formula for real N, and it is consistent with the exponential for $sgn\left(Re(n)\left[\frac{m}{2}\frac{\{x_{j+1}-x_j\}^2}{T}\right] - TRe\left[\frac{1}{n}\right]\int_0^1 V(x_j + (x_{j+1}-x_j)s)ds\right) = sgn\left(Re(n)\left[\frac{m}{2}\frac{\{x_{j+1}-x_j\}^2}{T}\right]\right)$ (and likewise for the imaginary part). The definition for the contribution formula is determined by extending the power series for the exponential to all V. The power series, the definition of the contribution formula, and the original exponential are exactly equal for arbitrarily small V.

$$\lim_{n\to\infty} K - \frac{i}{\hbar} B \int \int \ . \ . \ . \int \ exp\left(\frac{i}{\hbar}\mathrm{S}[0]\right)\left[exp\left(T\sum_{j=0}^{n-1}\int_0^1 V(x_j + (x_{j+1} - x_j)s)ds\right) - 1\right] dx_1 \ ... dx_{n-1} \ + \cdots$$

where the substitution of the exponential is accurate for small T. Furthermore, for such small time-translations, we may make the further simplification

$$\lim_{n\to\infty} K - \frac{i}{\hbar} B \int \int \ . \ . \ . \int \ exp\left(\frac{i}{\hbar}\mathrm{S}[0]\right)\left[exp\left(T\sum_{j=0}^{n-1} V(x_a)\right) - 1\right] dx_1 \ ... dx_{n-1} \ + \cdots$$

as the value of the potential V is nearly exactly $V(x_a)$, over the entire path, for small T. Following the general procedure of Section II, and equations A.1 and A.2, now with the appearance of the sum $\sum_{j=0}^{n-1} T\, V(x_a)$, we have

$$\lim_{n\to\infty} K - \frac{i}{\hbar} B(Kexp(nTV) - K) + \cdots$$

Which, for small T, yields

$$\lim_{n\to\infty} K - \frac{i}{\hbar} Bn(KTV) + \cdots$$

Producing[4]

$$K - \frac{i}{\hbar} DTVK + \cdots (10)$$

where V denotes the nearly uniform value of the potential along the infinitesimal path (equal to $V(x_a)$ above). In this case,

$$D = \exp(\mathrm{it})\{|\cos(t)|^p - |\sin(t)|^p\}\left(|\cos(t)|^p + |\sin(t)|^p\right)^{\frac{1}{p}-1} (11)$$

Given the formula (9).

B. The Generalized Schrodinger Equation

Relevant to the construction of the differential equation of the wavefunction is the behavior of the wavefunction over infinitesimal time-translations, and therefore the propagator over small T. Thus, let us consider the case of an arbitrarily small time-interval $T = \xi$. In this case, the approximations of 3.A are accurate, and the propagator is given by (10),

$$K - \frac{i}{\hbar} D\xi VK + O(\xi^2)$$

or,

$$Kexp\left(-\frac{i}{\hbar} D\xi V(x_a)\right)$$

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[4] The appearance of the n multiplying B is a result of (A. 1), which converts the sum over n terms V into the power n, producing $exp(nTV) - 1$ which resolves into nTV for small T.

Use of variable substitution, power series expansion, and the definition of the wavefunction in terms of the Kernel, as Feynman applies similarly in [2], suffices to demonstrate that

$$-\frac{\hbar^2}{2mz}\frac{\partial^2\psi}{\partial x^2}+DV\psi=i\hbar\frac{\partial\psi}{\partial t} \quad (12)$$

with z given as in (9) and D as in (11).

Explicitly, the generalized Schrodinger equation is given as

$$-\frac{\hbar^2}{2m|\exp(i\mu)|_p}\frac{\partial^2\psi}{\partial x^2}+\{|\cos(\mu)|^p-|\sin(\mu)|^p\}\,(|\cos(\mu)|^p+|\sin(\mu)|^p)^{\frac{1}{p}-1}V\psi=\exp(-\mathrm{i}\mu)\,i\hbar\frac{\partial\psi}{\partial t}$$

with the parameter μ running from π to $2\,\pi$.

This is the extended equation for the wavefunction, resulting from the generalized propagators that are derived from an analytic continuation of the path integral procedure. This is simply the standard Schrodinger Equation, but with a complex multiplier z appearing in front of the mass, and a factor D multiplying the potential. z is constrained to the particular region illustrated in **Fig. 1.** We, again, interpret these generalized versions of the Schrodinger Equation as corresponding to different universes with different associated laws of physics.

It should be noted that Feynman's process for deriving the Schrodinger Equation only applies in the case where z has a positive imaginary part.

IV. Difference with the Complex-Time Path Integral

In particle physics it is generally standard procedure to execute propagators over imaginary or complex lengths of time (e.g. [3]). The purpose of this is to ensure that certain mathematical expressions within the path integral are well-defined. We note that our unique analytic continuation procedure is entirely different from the mathematics of path integrals in imaginary or complex time. Rather than extending the time interval to the complex numbers, we are allowing the *index n itself*, a fundamental part of Feynman's formula, to be complex. Thus, rather than applying the basic laws of quantum mechanics to an imaginary length of time for the purpose of mathematical simplification, we have generalized *the laws of quantum theory themselves* and thereby produced a description of different universes where particles abide to these different laws.

Additionally, we see that our results are quite different from those of quantum theory in imaginary-time. We saw how our result for the free particle constituted a complex power of the standard propagator, with this complex exponent only appearing within a specific region in the complex plane as shown in **Fig. 1**. No such result is connected to path integration in imaginary time. The notion of the Euclidean action, which appears in path integration over an imaginary length of time, is well described in much scientific literature. We see that our equation, derived in (12), is noticeably different from that produced from the Euclidean action, in particular due to the D term, which modifies the role of the potential.

It is also worth noting that our generalized propagator (8) is equivalent to the standard equation, but with the mass M replaced by Mz. It is important to note that we still interpret our particles as having a mass M, but the wavefunctions of the particles evolve *as if* they had a mass Mz. Analysis of particles of complex mass has been used before in physics, notably in the modeling of certain unstable particles (e.g., see [1]). A particle with decay rate E may be *formally modeled* as having a "mass" M+iE, with M the mass in the usual sense. Our Mz formula is different, as it applies to *all* particles in a network of universes corresponding to different z, rather than just *some* unstable particles in one universe. Additionally, z is a complex number *in a specific region of the complex plane*, within the region described in **Fig. 1** (parameterized by (8)); the complex quantity M+iE, however, does not have this precise sort of constraint. Furthermore, of course, the D factor multiplying V in (12) yields particle behavior distinct from simply a complex mass, in the case of the presence of a potential. In conclusion, this multiverse hypothesis yields laws of physics very different from anything studied before.

Intriguingly, we could produce the same generalized equation (12) by certain manipulations involving complex time, namely by

1. *Considering a multiverse, in which the path integral contribution formula corresponding to each universe includes a complex number* w *multiplying the time interval T. All values of* w *together comprise this multiverse.*
2. *"Amending" this contribution formula, by taking the p-norm of the action, as in Section 1.B.*
3. *Multiplying the mass, within the contribution formula, by* w*, and the potential V by 1/*w*. Each different contribution formula yields different particle propagators, producing the generalized Schrodinger equation appearing throughout the multiverse as expressed in (12).*

This process for producing a multiverse whose physical law is delineated by (12) does not require the use of a complex discretization index. Nonetheless, the steps (1) – (3) are entirely arbitrary and unmotivated, whereas our generalization of path skeletonization is a natural extension of quantum mechanics. The consideration of all complex scalings of the time interval, as in step (1), is unmotivated, whereas the consideration of different limits to infinity of the skeletonization index constitutes a natural extension of path integration. Furthermore, the introduction of the p-norm as in step (2), in order to protect the nature of the path integrand as a phase factor in the midst of the generalization, is a novel conception of the present paper. Crucially, however, the multiplication of the mass by *w* and the potential by 1/*w* is a unique property of the present generalization. The effective mass multiplication is caused by the process of integration over a complex number of variables, which can produce a complex result even if the integrand is real, as in (A.1). The 1/*w* term is a result of the fact that the sum containing the potential consists of a complex number of terms; the integration over a complex number of variables, and the computation of the resulting limit, produce a complex result, even though the original action is interpreted to be real, as in (7).

V. Separation of the Generalized Schrodinger Equation and Unitarity

In order to obtain useful physical models for nonrelativistic particle behavior, it is standard to separate the Schrodinger Equation so as to derive the time-independent Schrodinger Equation as well as the fundamental time-dependence expressions. We shall apply the same process here. Given (12),

$$-\frac{\hbar^2}{2mz}\frac{\partial^2\psi}{\partial x^2}+DV\psi=i\hbar\frac{\partial\psi}{\partial t}$$

we set the wave function as an expression of the form X(x)T(t), with X,T functions of space and time, respectively. This will allow us to determine the basic separated solutions, and permit us to express other wave functions as combinations of these solutions. Multiplying through by z, inputting our form for the solutions, dividing by XT, and setting both sides equal to a constant, we find

$$-\frac{\hbar^2}{2m|\exp(i\mu)|_p}\frac{\partial^2 X}{\partial x^2}+\{|\cos(\mu)|^p-|\sin(\mu)|^p\}\left(|\cos(\mu)|^p+|\sin(\mu)|^p\right)^{\frac{1}{p}-1}VX=EX$$

as our time-independent equation and

$$T(t)=\exp\left(-\frac{i}{\hbar}\mathrm{E}\exp(i\mu)\,t\right)\quad(13)$$

as our time-dependence expression, with μ again constrained between π and 2π, and E a constant.

The time-dependence term (13) immediately yields issues with unitarity, as no normalization constant could ensure the conservation of probability. While this may seem a severe problem of the model, we argue that conservation of probability is *not*, in general, required for physical interpretation. For example, suppose 100 particles are released independently in a scientific apparatus. In “normal” quantum mechanics, we would calculate a 100 percent probability that we would find every particle *somewhere* in space, as “normal” quantum models are generally unitary. However, our time-dependence factor, which decays over time, clearly shows that our states cannot be normalized, generally speaking. This is not meaningless; in our scenario, it means that some of the 100 particles *disappear* over time. If probabilities add up to, say, thirty percent, this means that we will find on average about thirty particles at the end of the experiment. Thus, because our probabilities generally decay over time, this means that particles have the capacity to vanish completely. This is an unusual property, but is *not* physically meaningless, and we could imagine the present multiverse model as logically consistent, despite the lack of unitarity in most universes.

Conclusions and Possible Future Research

We generalize the process of path integration in quantum mechanics to the case of a complex-valued time-discretization index. Subject to an augmented definition of the action, to maintain the status of the contribution formula as a pure phase factor, this generalization leads to multiple different possible propagators, due to the fact that the limit at infinity of the index is not unique. These different propagators are postulated to apply to different universes. In this framework, the solution for the free particle, as well as a particle in a potential, is analyzed, while a generalized Schrodinger Equation is developed. The differences between this approach, and path integration in imaginary time, is emphasized, as this generalization involves a complex-valued time-slicing index as opposed to time-interval. Additionally, we offer a physical interpretation of this multiverse model, consistent with its non-unitary nature.

Recent research in theoretical physics and cosmology has been founded on mathematical generalizations of physical law, notably including the anthropic analysis of the cosmological constant, which sees an extension of the Einstein field equations within the context of a multiverse model [8]. Additional such research has also focused on the implications of superstring theory, especially with respect to an anthropic landscape [4]. This model, which offers a natural analytic continuation of the Feynman path integral to complex time-slicing index, as well as an analogous "landscape" interpretation, could also pose applications to other fields of physics as well.

In the future, we would like to see an extension of the present model to relativistic physics, especially utilizing the path integral formulation of quantum field theory. Although such an approach would see difficulties in mathematical formulation, due to the measure problem associated with functional integrals in relativistic quantum theory, this extension could nonetheless provide novel information on the relativistic behavior of the multiverse model. Additionally, an extension of the model to quantum field theory could also provide enlightenment on its non-unitary nature [10]. Functional integrals associated with QFT, as demonstrated by Edward Witten and others [9], could find support as topological invariants in the fields of knot theory and low-dimensional topology; an analytic continuation of functional integration to complex discretization indices, as per the techniques of the present paper, and an associated interpretation within an anthropic multiverse context, could possibly prove useful to these domains of mathematics and lend additional support to this physical model. Furthermore, a cosmological anthropic analysis of universe evolution within this multiverse model could provide, along with additional theoretical support, possible experimental application [7].

Appendix: Mathematical Considerations

In the course of ascertaining the values of certain integrals over a complex number of variables, and determining the analytic continuation of path integration to a complex form of path skeletonization, we have used a number of assumptions about the behavior of certain expressions that are not explicitly defined. The purpose of the present appendix is *not to provide a mathematically rigorous formulation* of this continuation (which appears impossible, as none of these expressions can be explicitly delineated, when subject to a complex number of terms or variables), but to provide a consistent framework for calculating quantities relevant to path integration in the case of complex N. We might view this enterprise as similar in spirit, for example, to the notion of calculating derivatives of fractional order, but with necessarily less mathematical rigor. Ultimately, to effect an analytic continuation of this intricacy, we must decide which properties to preserve from the case of real N, and which to discard; we shall see that preserving the additive nature of integrals requires forfeiting some aspects of the real case. This particular framework, built up from a few postulates concerning the behavior of these expressions, is derived from natural assumptions about the nature of certain expressions, generalized from the real case.

The Dirac delta function, although not explicitly defined at every point in its domain, does possess a well-defined integral. Similarly, although the action (and its exponential) is a "function" of a complex number of variables in the case of complex N, and thus not truly defined, its integral is. Our key postulate in this regard is the *separation of variables:*

$$\int\int \ldots \int \prod_{j=1}^{n} f(x_j)\, dx_1 \ldots dx_n = \left(\int f(x)\, dx\right)^n \quad (\mathrm{A.1})$$

This separation allows us to determine an explicit value from the integral of a "function" that is not truly explicitly defined[5]. As such, as is clear in the computation of the free particle propagator, it allows us to determine a real result for a physical quantity.

In addition, as seen in Section II, we also postulate that variable substitution is possible in such multiple integrals; in particular, due to the importance of this type of integral in the physical theory, we postulate that

$$\int \ldots \int f\left(\sum_{j=1}^{n} g(x_j - x_{j-1})\right) dx_1 \ldots dx_{n-1} = \int \ldots \int f\left(\sum_{j=1}^{n-1} g(z_j) + g\left(x_n - x_0 - \sum_{j=1}^{n-1} z_j\right)\right) dz_1 \ldots dz_{n-1} \quad (\mathrm{A.2})$$

[5] This formula, in addition to allowing for the free-particle propagator to be determined, also serves to reduce the potential perturbation expansion to a calculable form, producing the exp(it) factor appearing in D (deriving from the transformation from a *sum* over a complex number n of terms, each being the value of the potential V, to a *power* of n, as a result of the integral formula).

provided that the limits of integration are infinite, and that f and g are explicitly defined analytic functions; this type of variable substitution is clearly allowed in the real case, and the specific integrals involved are particularly important to the physical theory.

These postulates, generalized to the complex n, constitute natural extensions of properties clearly true for integer n. In addition to postulating that the real or imaginary part of a sum is the respective sum of the real or imaginary parts of the terms, we also define the p-magnitude as

$$|A|_p = \left(\sqrt{(Re(A))^2}^p + \sqrt{(Im(A))^2}^p\right)^{1/p} \quad (A.3)$$

Here, we assume that any sum of positive real terms is positive[6], and that $\sqrt{\mu^2}$ is equal to μ for μ positive. In addition, to effect the construction of the physical theory, we must determine a distributive property of complex-order sums, namely that

$$\sum_{j=1}^{n} a \cdot g_i + b \cdot h_i = a \cdot \sum_{j=1}^{n} g_i + b \cdot \sum_{j=1}^{n} h_i \;\; (A.4)$$

for a, b explicitly defined constants[7].

The integration postulate (A.1) (along with the postulate concerning variable substitution, expressed in (A.2)), the summation postulate (A.4), the definition of the general p-norm in (A.3), notions of positivity, the linear nature of real and imaginary parts, and the notion that analytic functions $f(z)$ can be expanded in a Fourier integral representation (and that the exponential of a sum is the product of the exponentials) as in Section II, are the only postulates needed to define the natural analytic continuation of path integration to complex discretization index. As such, although a variety of unusual properties concerning real parts and integrals were overviewed in this Appendix, all arise from these simple postulates. (A.1) particularly, along with a few reasonable assumptions about limits[8], serves to define the value of the Feynman path integral in the case of a free particle, as well as the form of the generalized Schrodinger equation, as in Sections II and III.

Analytic continuations like the present case are not unknown in theoretical physics, particularly in the domain of quantum field theory. As is well known, the calculation of the Casimir Force between two metal plates rests upon a perturbation expansion, with the terms diverging as in $\sum_{i=0}^{\infty} i$; by generalizing the behavior of the Riemann zeta function known to hold for converging series, to the case of the

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[6] If we assume that we can "separate" any sum as in $\sum_{j=1}^{z-1} a_j = \sum_{j=1}^{n} a_j + \sum_{j=1}^{z-n-1} a_{j+n}$, for n integer, we arrive at a contradiction, for example, in the case that a_j is a positive constant, $z-1$ is a positive integer, and $z-n-1$ is negative, as this relation assigns a negative value to $\sum_{j=1}^{z-n-1} a_{j+n}$, a sum of positive terms. Thus, to maintain consistency, we cannot assume that the above relation works when z is complex; that is, the first or last integer number of terms cannot be separated from the sum. Indeed, there is no reason to expect that it is even meaningful to refer to the "first" term of a complex number of summands

[7] We also postulate that $Re\left(\sum_{i=1}^{n} a_j\right) = \sum_{i=1}^{n} Re\left(a_j\right)$, and similarly for the imaginary part, as described above

[8] Namely, that limits are linear, permitting the decomposition of Section III into terms of different order in T. Also, we must make the obvious assumption that limits of summation can be re-indexed as in $\sum_{i=0}^{z} a_i = \sum_{i=n}^{z+n} a_{i-n}$ for integer n.

divergent expansion, it was concluded that the result was actually *negative*, leading to an expression for the force confirmed by experiment [6]. Although such reasoning was certainly by no means rigorous, it was eventually placed on a mathematically accurate foundation. Similarly, the present analytic continuation generalizes the behavior of integrals, sums, and real parts, as relevant to physical systems, to the case of complex n, producing results (like an integral of a real-valued function being complex) that are as unusual as a divergent series of positive terms producing a negative "result." In both cases, an analytic continuation serves to generalize known relations to a new realm, where unusual phenomena occur; the simplest aspect of the analytic continuation is present in (A.1).

These results seem strange, but sums of a complex number of terms are expected to behave in ways radically different from "explicitly defined" numbers. Indeed, it is impossible to ascribe a particular value to any of these expressions, and we therefore must use generalizations of the behavior of "usual" numbers to determine the mechanics of applying real or imaginary parts, or integrals over a complex number of variables. In a vague sense, we might view these expressions as "fuzzy," without an explicitly defined value, and the act of integration, as in (A.1), serves to establish an actual value. This is not unlike the case of the Dirac delta function, which, although undefined over its domain, still yields an explicitly defined integral.

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